Homework & Solution

1. Sec. 5.4 Q19

19. Let A denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix},$$

where $a_0, a_1, \ldots, a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of A is

$$(-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k).$$

Hint: Use mathematical induction on k, expanding the determinant along the first row.

- For $k=2$ $\quad A = \begin{pmatrix} 0 & -a_0 \\ 1 & -a_1 \end{pmatrix}$

$$f_A(t) = \det(A - tI_2) = \det\left(\begin{pmatrix} -t & -a_0 \\ 1 & -a_1-t \end{pmatrix}\right) = (-1)^2(a_0 + a_1 t + t^2)$$

- Suppose this is true for $k-1$
- If A is a $k\times k$ matrix

$$f_A(t) = \det(A - tI_k) = \begin{vmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -t & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -t-a_{k-1} \end{vmatrix}$$

$$= (-t)\cdot \begin{vmatrix} -t & \cdots & 0 & -a_1 \\ 1 & \cdots & 0 & -a_2 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & -t & -a_{k-2} \\ 0 & \cdots & 1 & -t-a_{k-1} \end{vmatrix} + (-1)^{1+k}(-a_0) \begin{vmatrix} 1 & -t & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -t \\ 0 & 0 & \cdots & 1 \end{vmatrix}$$

by assumption

$$= (-t)\,(-1)^{k-1}\cdot(a_1 + a_2 t + \cdots + a_{k-1}t^{k-2} + t^{k-1}) + (-1)^k\cdot a_0$$

$$= (-1)^k(a_0 + a_1 t + a_2 t^2 + \cdots + a_{k-1}t^{k-1} + t^k)$$

2. Sec. 5.4 Q24

24. Prove that the restriction of a diagonalizable linear operator T to any nontrivial T-invariant subspace is also diagonalizable. *Hint:* Use the result of Exercise 23.

Since T is diagonalizable.

$V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ where $\lambda_1 \cdots \lambda_k$ are distinct eigen values.

Then $W = W \cap V = W \cap E_{\lambda_1} \oplus W \cap E_{\lambda_2} \oplus \cdots \oplus W \cap E_{\lambda_k}$

Let $\beta_j = \{v_1^j \cdots v_{n_j}^j\}$ be a basis for $W \cap E_{\lambda_j}$, $j = 1 \cdots k$

Let $\beta = \bigcup_{j=1}^{k} \beta_j$

Claim: β is a basis for W.

① β is L.I. Consider

$$\sum_{j=1}^{k} \sum_{l=1}^{n_j} a_l^j v_l^j = 0$$

since $v^j = \sum_{l=1}^{n_j} a_l^j v_l^j \in W \cap E_{\lambda_j}$

v^j is an eigen vector of T corresponding to eigen value λ_j

and $v^1 + \cdots + v^k = 0 \in \{0\}$ the trivial T-invariant subspace.

By exercise 23. we have $v^j \in \{0\}$ for $j = 1, \cdots, k$

Since $\{v_1^j \cdots v_{n_j}^j\}$ is basis for $W \cap E_{\lambda_j}$, thus L.I.

we have $a_l^j = 0$ for $l = 1 \cdots n_j$, for $j = 1, \cdots, k$

② β span W

$$\begin{aligned} W = W \cap E_{\lambda_1} \oplus \cdots \oplus W \cap E_{\lambda_k} &= \text{span}(\beta_1) + \cdots + \text{span}(\beta_k) \\ &= \text{span}(\beta_1 \cup \cdots \cup \beta_k) \\ &= \text{span}(\beta) \end{aligned}$$

3. Sec. 5.4 Q28

28. Let $f(t)$, $g(t)$, and $h(t)$ be the characteristic polynomials of T, $\mathsf{T_W}$, and $\overline{\mathsf{T}}$, respectively. Prove that $f(t) = g(t)h(t)$. *Hint:* Extend an ordered basis $\gamma = \{v_1, v_2, \ldots, v_k\}$ for W to an ordered basis $\beta = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V. Then show that the collection of cosets $\alpha = \{v_{k+1} + \mathsf{W}, v_{k+2} + \mathsf{W}, \ldots, v_n + \mathsf{W}\}$ is an ordered basis for $\mathsf{V/W}$, and prove that

$$[\mathsf{T}]_\beta = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix},$$

where $B_1 = [\mathsf{T}]_\gamma$ and $B_3 = [\overline{\mathsf{T}}]_\alpha$.

$\gamma = \{v_1 \cdots v_k\}$ is basis for W.

extend γ to $\beta = \{v_1 \cdots v_k, v_{k+1} \cdots v_n\}$ a basis for V.

Let $\alpha = \{v_{k+1} + W, \cdots, v_n + W\}$

Claim: α is a basis for V/W

① α is L.I.

Consider $a_{k+1}(v_{k+1}+W) + \cdots + a_n(v_n+W) = W$

i.e. $a_{k+1}v_{k+1} + \cdots + a_n v_n \in W = \text{span}(\gamma)$

$\exists\ a_1 \cdots a_k \in F$ st $a_{k+1}v_{k+1} + \cdots + a_n v_n = a_1 v_1 + \cdots + a_k v_k$

since β is L.I. we have $a_1 = \cdots = a_n = 0$

So α is L.I.

② $\forall\ v+W \in V/W$. $v \in V$.

$\exists\ a_1 \cdots a_n \in F$ st $v = a_1 v_1 + \cdots + a_n v_n$

Then $v+W = (a_1 v_1 + \cdots + a_n v_n) + W$

$= a_1 \cdot (v_1 + W) + \cdots + a_n \cdot (v_n + W)$

$= a_{k+1}(v_{k+1} + W) + \cdots + a_n \cdot (v_n + W)$

$\in \text{span}(\alpha)$

$$[T]_\beta = \begin{pmatrix} [T(v_1)]_\beta & \cdots & [T(v_n)]_\beta \end{pmatrix}$$

$$= \begin{pmatrix} [T_W(v_1)]_\gamma & \cdots & [T_W(v_k)]_\gamma & * & \cdots & * \\ 0 & \cdots & 0 & [\bar{T}(v_{k+1}+W)]_\alpha & \cdots & [\bar{T}(v_n+W)]_\alpha \end{pmatrix}$$

$$= \begin{pmatrix} [T_W]_\gamma & * \\ O & [\bar{T}]_\alpha \end{pmatrix}$$

Thus

$$f(t) = \det([T]_\beta - tI) = \det([T_W]_\gamma - tI) \cdot \det([\bar{T}]_\alpha - tI)$$

$$= g(t) \cdot h(t)$$

4. Sec. 5.4 Q30

30. Prove that if both T_W and $\overline{\mathsf{T}}$ are diagonalizable and have no common eigenvalues, then T is diagonalizable.

- T_W is diagonalizable, then $\exists\ \beta = \{v_1 \cdots v_k\}$ basis for W,

  s.t. $[T_W]_\beta = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix}$ i.e. $T(v_i) = T_W(v_i) = \lambda_i v_i \quad i = 1, \cdots, k$

- $\overline{T}$ is diagonalizable, then $\exists\ \gamma = \{u_{k+1} + W, \cdots, u_n + W\}$ basis for V/W

  s.t. $[\overline{T}]_\gamma = \begin{pmatrix} \lambda_{k+1} & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ i.e. $T(u_j) + W = \overline{T}(u_j + W) = \lambda_j(u_j + W) = \lambda_j u_j + W$

  $j = k+1 \cdots n$

- Since $\lambda_j u_j - T(u_j) \in W$, for $j = k+1, \cdots, n$

  Consider $T_W - \lambda_j \cdot I_W : W \to W$ for $j = k+1, \cdots, n$

  Since T_W and $\overline{T}$ have no common eigen values, $\lambda_i \neq \lambda_j$ $\quad i = 1 \cdots k,\ j = k+1 \cdots n$

  All eigen values $\lambda_1 - \lambda_j, \cdots, \lambda_k - \lambda_j$ of $T_W - \lambda_j I_W$ are non-zero

  Thus $T_W - \lambda_j I_W$ is invertible.

  $\exists\ w_j \in W$, s.t. $T_W(w_j) - \lambda_j w_j = (T_W - \lambda_j I_W)(w_j) = \lambda_j u_j - T(u_j) \in W$

- define $v_j = u_j + w_j \in V$ for $j = k+1, \cdots, n$

  $T(v_j) = T(u_j) + T(w_j) = T(u_j) + \lambda_j w_j + \lambda_j u_j - T(u_j)$

  $= \lambda_j(u_j + w_j) = \lambda_j v_j$.

  i.e. v_j is eigen vector of T corresponding to λ_j for $j = k+1 \cdots n$

- Consider $a_1 v_1 + \cdots + a_k v_k + \cdots + a_n v_n = 0 \quad ☆$

  $\underbrace{(a_1 v_1 + \cdots + a_k v_k)}_{\in W} + (a_{k+1} u_{k+1} + \cdots + a_n u_n) + \underbrace{(a_{k+1} w_{k+1} + \cdots + a_n w_n)}_{\in W} = 0$

  then $a_{k+1} u_{k+1} + \cdots + a_n u_n \in W$

i.e $a_{k+1}(u_{k+1}+W)+\cdots+a_n(u_n+W)=W$

Since γ is a basis for V/W. thus is L.I

we have $a_{k+1}=\cdots=a_n=0$

Then $\star$ becomes $a_1v_1+\cdots+a_kv_k=0$

since β is basis for W, we have $a_1=\cdots=a_k=0$

Thus $\{v_1 \cdots v_n\}$ is L.I.

since $|\{v_1\cdots v_n\}| = n = \dim(V)$

$\{v_1\cdots v_n\}$ is a basis for V consisting eigen vectors of T.

Thus T is diagonalizable

5. (Tensor Product) Define the trace of a linear operator T as the trace of its matrix representation under any basis, i.e. $tr(T) := tr([T]_\beta)$.

Let V and W be finite-dimensional spaces. Let $T : V \to V$ and $U : W \to W$ be linear. Define a mapping $T \otimes U : V \otimes W \to V \otimes W$ by $T \otimes U(v \otimes w) = T(v) \otimes U(w)$.

Show that $tr(T \otimes U) = tr(T)tr(U)$.

Remark: $tr(T) := tr([T]_\beta)$ is well-defined since it's independent of choice of basis β.

$T : V \to V \qquad U : W \to W$

$T \otimes U : V \otimes W \to V \otimes W$

$v \otimes w \mapsto T(v) \otimes U(w)$

Let $\beta = \{v_1 \cdots v_n\}$ be basis for V

$\gamma = \{w_1 \cdots w_m\}$ be basis for W.

Let $[T]_\beta = A$. then $T(v_j) = \sum_{i=1}^{n} A_{ij} v_i$

Let $[U]_\gamma = B$, then $U(w_k) = \sum_{l=1}^{m} B_{lk} w_l$

$$\begin{aligned}
T \otimes U (v_j \otimes w_k) &= T(v_j) \otimes U(w_k) \\
&= \left(\sum_{i=1}^{n} A_{ij} v_i\right) \otimes \left(\sum_{l=1}^{m} B_{lk} w_l\right) \\
&= \sum_{i=1}^{n} \sum_{l=1}^{m} A_{ij} \cdot B_{lk} \cdot v_i \otimes w_l
\end{aligned}$$

$$\begin{aligned}
tr(T \otimes U) &= \sum_{j=1}^{n} \sum_{k=1}^{m} \left(\sum_{i=1}^{n} \sum_{l=1}^{m} A_{ij} B_{lk} \cdot \delta_{ij} \delta_{kl} \right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{m} A_{jj} B_{kk} \\
&= \left(\sum_{j=1}^{n} A_{jj}\right)\left(\sum_{k=1}^{m} B_{kk}\right) \\
&= tr(A) \cdot tr(B) \\
&= tr([T]_\beta) \cdot tr([U]_\gamma) \\
&= tr(T) \cdot tr(U)
\end{aligned}$$